FAST BETATRON TUNE CONTROLLER FOR CIRCULATING BEAM IN A SYNCHROTRON

Takuyuki ENDO, Kichiji HATANAKA, Kenji SATO, Takane SAITO, Shinji ANO, Hitoshi TAMURA, Kouji NODA*, Mitsutaka KANAZAWA, Satoru YAMADA, Hiromitsu INOUE**, Osamu TAKEDA** and Masataka MIZOBATA**
Research Center for Nuclear Physics, Osaka Univ.,
Osaka 567, Japan,
*National Institute of Radiological Sciences,
Chiba 263, Japan,
**Energy and Industrial Systems Center, Mitsubishi Electric Corporation,
Kobe 652, Japan

Abstract

When rf quadrupole(RFQ) electric field is applied to the circulating beam in a synchrotron, an equation of motion is reduced to *Mathieu's Equation*. A new analytical method to obtain an approximate solution has been developed, while a numerical computation was usually applied. Translating the behavior of approximate solution into terms of an RFQ electric field and betatron oscillation, a fast tune controll can be achieved by rapid tuning of both amplitude and frequency of rf voltage. This process could be applied to suppress a tune shift caused by a space charge effect and to controll a slow beam extraction with a low ripple. We have started another analytical computation using Hamiltonian with perturbation of RFQ and the results of this computation also suggest that it is applicable to slow beam extraction. The fast tune controller has been constructed and the beam test will be performed at HIMAC synchrotron in cooperation of RCNP and NIRS.

1 Introduction

An rf knockout slow extraction method [1] has been developed at HIMAC synchrotron constructed at the National Institute of Radiological Sciences(NIRS). This process utilizes a transverse rf electric field resonated with horizontal betatron tune, where horizontal and vertical kicker electrode is applied to the beam in a synchrotron. Thus, a similar slow extraction method can be expected by adopting RF Quadrupole(RFQ) electric field.

Although RFQ is usually used as a linear accelerator in the low-energy range, we studied applying it to the circulating beam in a synchrotron. When RFQ electrodes have no modulation, a field potential of RFQ corresponds to that of quadrupole magnet whose current varies periodically at radio-frequency. Since a betatron tune value of beam depends on the strength of quadrupole magnet in the ring, a betatron tune value is modified due to RFQ electric field. If the stability of motion in a synchrotron is preserved, some variations of betatron frequency can be expected.

In this paper, we introduce an equation of the motion which is given by Mathieu's Equation, and describe a new analytical method to obtain the approximate solution of the equation.

2 An analytical expression of Mathieu's equation

2.1 The equation of motion for rf quadrupole

When RFQ electric field is applied to the beam in a synchrotron, an equation of motion is given by:

$$\frac{d^2x}{ds^2} + K_0(s)x = \frac{q_e}{m_0\gamma v^2}E_x(s,x,t), \tag{1}$$

where $s(=vt)$ is a longitudinal distance, and $K_0(s)$ is a periodic function of s.

We can express the effect of RFQ electric field by summation of step function $u(s)$ as given by:

$$\begin{aligned} E_x(s,x,t) &= G_0 x \sin(\omega t + \Phi) \\ &\times \sum_{n=-\infty}^{\infty} [u(s+\frac{l}{2}+nL) - u(s-\frac{l}{2}+nL)], \end{aligned} \tag{2}$$

where G_0 is a field gradient; l is an RFQ electrode length; and L is a circumference of the ring.

Assuming that

$$\begin{aligned} &A\cos(K\phi + \Phi_k) \\ &= \sin(\omega t + \Phi) \sum_{n=-\infty}^{\infty} [u(s+\frac{l}{2}+nL) - u(s-\frac{l}{2}+nL)], \end{aligned} \tag{3}$$

the equation of motion is reduced to Mathieu's Equation as given by:

$$\frac{d^2y}{dz^2} + a\{1 - \frac{2q}{a}\cos(2z)\}y = 0, \tag{4}$$

where

$$\begin{cases} y = \dfrac{x}{\sqrt{\beta_0}} & ,2z = K\phi + \Phi_k, \\ a = \dfrac{4}{K^2}\nu_0^2, & ;\phi = \dfrac{1}{\nu_0}\displaystyle\int_0^s \frac{ds}{\beta_0} \\ \dfrac{2q}{a} = \dfrac{q_e}{m_0\gamma v^2}G_0\beta_0^2 A & \end{cases}$$

;and ν_0 is a betatron tune which depends on β_0.

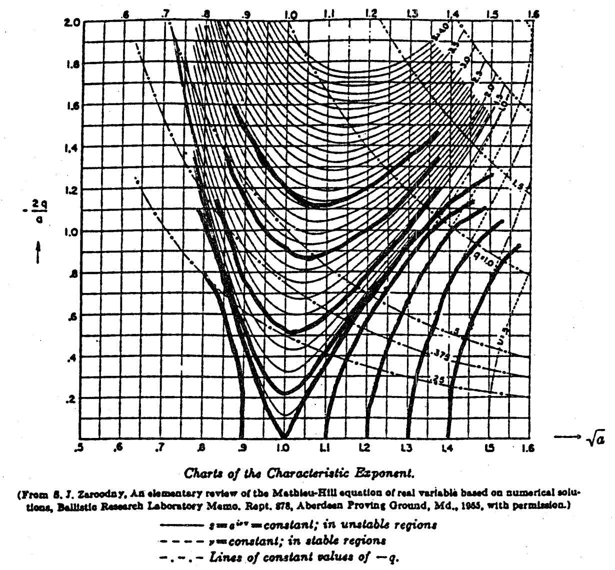

Fig. 1 Comparison between numerical solution and analytical one. Bold lines correspond to lines of constant values of ν.

2.2 Mathieu's Equation

Since the coefficients of Mathieu's equation

$$\frac{d^2y}{dz^2} + a\{1 - \frac{2q}{a}\cos(2z)\}y = 0 \tag{5}$$

are periodic functions of z, it follows from *Floquet's Theorem* that there exists a solution of the form

$$F_\nu(z) = e^{i\nu z}P(z),$$

where ν depends on a and q, and is called the *characteristic exponent*; $P(z)$ is a periodic function, of the same period as that of the coefficients in Eq.(5).

A numerical solution of Mathieu's equation has been obtained [2] for each *characteristic exponent* as shown in Fig.1 (from HANDBOOK OF MATHEMATICAL FUNCTIONS).

For small q, the following expansions may be used

$$F_\nu(z) = c_0\left[e^{i\nu z} - q\left\{\frac{e^{i(\nu+2)z}}{4(\nu+1)} - \underline{\frac{e^{i(\nu-2)z}}{4(\nu-1)}}\right\}\right] + \cdots, \tag{6}$$

where ν is not an integer. If the *characteristic exponent* $\nu \sim 1$, the term with underline, which has a frequency of $(2-\nu) \sim 1$, in the above expansions has main contribution. Utilizing this frequency, we develop a new analytical study in the next section.

2.3 A new analytical method for Mathieu's equation

Taking conjugate momentum $p = \frac{1}{\sqrt{a}}y'$, we transform y and p into r and θ in Mathieu's equation as given by:

$$\begin{cases} y = r\cos(Qz - \theta) \\ p = -r\sin(Qz - \theta), \end{cases} \tag{7}$$

where Q is a resultant frequency of oscillation.

Assuming that $Q \sim 1 (Q = 2 - \nu)$ as derived from previous section, we remain only slowly-varying terms and neglect rapidly oscillating components which average to zero. The Mathieu's equation can be approximated by

$$\begin{cases} r' = -\frac{1}{4}\sqrt{a}\left(-\frac{2q}{a}\right) r\sin\{(2-2Q)z + 2\theta\} \\ \theta' = -(\sqrt{a} - Q) - \frac{1}{4}\sqrt{a}\left(-\frac{2q}{a}\right)\cos\{(2-2Q)z + 2\theta\}. \end{cases} \tag{8}$$

Transforming θ into ψ, where $\psi = (1-\theta)z + \theta$, we obtain approximate solution as given by:

$$\begin{cases} \tan\psi = -\sqrt{\frac{A+B}{A-B}}\tan(\sqrt{A^2 - B^2}z + k) \\ r^2 = \frac{C}{A^2 - B^2}\{A + 2B\cos(2\sqrt{A^2 - B^2}z + k)\}, \end{cases} \tag{9}$$

where $|A| > |B|$(stable region),

$$\begin{cases} A = \sqrt{a} - 1 \\ B = \frac{1}{4}\sqrt{a}\left(-\frac{2q}{a}\right). \end{cases}$$

Similarly, in unstable region ($|A| < |B|$),

$$\begin{cases} \tan\psi = \sqrt{\frac{B^2 + A^2}{B^2 - A^2}}\coth(\sqrt{B^2 - A^2}z + k) \\ r^2 = \frac{C}{2(B^2 - A^2)}\{A + B\cosh(\sqrt{B^2 - A^2}z + k)\}. \end{cases} \tag{10}$$

2.4 Comparison between numerical solution and analytical one

Assuming that frequency of oscillation in Eq.(9) corresponds to that in Eq.(8), the expression which represents the realtion between parameter q, a and Q in stable region is obtained as given by:

$$2\sqrt{A^2 - B^2} = 2 - 2Q \qquad (Q = 2 - \nu)$$

$$\Rightarrow -\frac{2q}{a} = 4\frac{\sqrt{(\sqrt{a}-1)^2 - (Q-1)^2}}{\sqrt{a}}. \tag{11}$$

The expression in unstable region is also obtained as given by:

$$\pi\sqrt{B^2 - A^2} = lnS \qquad (S = e^{i\nu\pi})$$

$$\Rightarrow -\frac{2q}{a} = 4\frac{\sqrt{(\sqrt{a}-1)^2 + (\frac{1}{\pi}lnS)^2}}{\sqrt{a}}, \tag{12}$$

where ν is complex.

Fig.1 shows comparison between numerical solution and analytical one. In stable region, analytical expression corresponds to a line of constant value of ν, where $Q = 2 - \nu$. In unstable region, it corresponds to a lines of $S = e^{i\nu\pi}$ =constant.

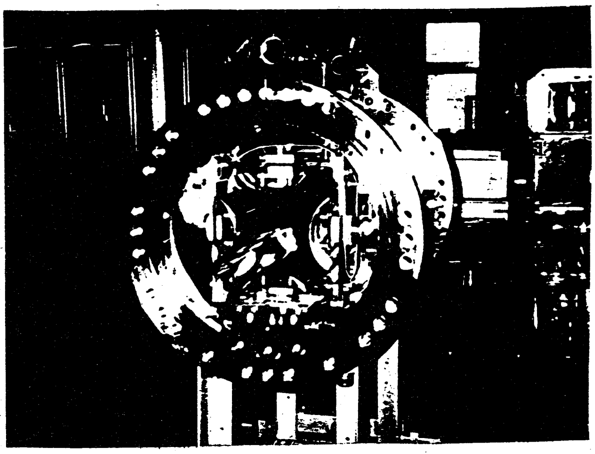

Fig. 2 A photograph of the fast tune controller

3 Design and example of tune shift

With an analytical solution of r and θ, we can express a particle motion as given by:

$$x = c_0\sqrt{\beta_0}\eta = c_0\sqrt{\beta_0}r\cos\left(Q\frac{\nu_0}{\sqrt{a}}\phi - \theta + \frac{1}{2}\Psi_k\right), \tag{13}$$

where c_0 is constant.

From Eq.(11), a variation of q for a fixed a corresponds to a variation of Q, where parameter q is proportional to the amplitude of RFQ electric field; and parameter a is related with frequency of RFQ and that of revolution as given by:

$$f_{rfq} = f_{rev}(\frac{2\nu_0}{\sqrt{a}} - m) \qquad (m = 0, \pm 1, \pm 2, \cdots), \tag{14}$$

where

$$f_{rfq} = \frac{\omega}{2\pi}, f_{rev} = \frac{v}{L}.$$

Eq.(14) is derived from the relation between parameters in the eq.(3). In this case, the original tune ν_0 is shifted to the resultant tune $Q\frac{\nu_0}{\sqrt{a}}$.

Table 1 shows an example of tune shift for vertical tune, where the experiment is assumed to be performed at HIMAC synchrotron($L = 130[m]$).

The fast betatron tune controller has been constructed and the high power performance test was done up to 1 kW rf power. The rf voltage over 1.4 kVpp was obtained at 1 kW rf power in the range from 0.3 MHz to 2.0 MHz using the bridged-T type all-pass network. In the next stage, the authors plan to install this controller in HIMAC, and the beam test will be performed soon. Fig.2 shows the photograph of the fast tune controller. This controller has the same design as shown in Table 1.

Table 1
An example of tune shift for vertical tune at HIMAC synchrotron

original tune	$\nu_0 = 3.13$
resultant tune	$\nu = 3.123$
value of tune shift	$\Delta\nu = 0.007$
ratio between the frequency of RFQ and that of revolution	$\frac{f_{rfq}}{f_{rev}} = 2.215$
energy par nucleon	$6[MeV/u]$
ratio between charge and mass	$\varepsilon = 1/2$
β function	$\beta_0 = 15.4[m]$
bore radius	$65[mm]$
length of electrode	$0.6[m]$
field gradient	$426[kV/m^2]$
voltage	$1.8[kV_{pp}]$
RF Power	$1.0[kW]$

4 Conclusion and discussion

An analytical solution of Mathieu's equation, which describes a motion of circulating beam perturbed by RFQ, was obtained. From this solution, a fast tune controll can be achieved by rapid tuning of both amplitude and frequency of rf voltage. This process could be applied to suppress a tune shift caused by a space charge effect, although the ordinary tune shift by quadrupole magnet is not applicable because of its slow response.

On the other hand, our computation of transfer matrix which had an RFQ strength varying with every turn suggests that the betatron tune value oscilates itself with every turn and its mean value is shifted from the original one. From this result, the motion of beam perturbed by RFQ is recognized that it includes a tune shift with some tune spread. This behavior corresponds to the oscilation of r in the eq.(9), (10) and (13). Thus we can utilize it to controll a slow beam extraction with a low ripple.

Furthermore, We have started another analytical computation using Hamiltonian with perturbation of RFQ. So far this results also suggest that RFQ is applicable to slow beam extraction.

These are the problems which need more studies.

References

[1] K. Noda *et al*, *Nucl. Instrum. Methods* A **374**, 269 (1996).

[2] HANDBOOK OF MATHEMATICAL FUNCTIONS, ed. M.Abramowitz and I.A.Stegun(Dover Publications, Inc., New York, 1970).