A New Two Particle Model for Study of Effects of Space-Charge Force on Beam Instabilities*.

Yong Ho Chin (KEK), Alex Chao (SLAC)
and Mike Blaskiewicz (BNL)

The 12th Annual Meeting of PASJ
Tsuruga, Japan, on 5 - 7August, 2015

*Paper Submitted to PRST-AB for Publication.

Outline

- Motivations of This Work
- Chao's Original Two Particle Model
- New Two Particle Model with Space-Charge
- Procedure to Identify Unstable Regions and to Compute Growth Rate
- Findings and Conclusions

No Beam Instability Observed at RCS

- No beam instability has been observed at RCS.
- It is generally believed that a beam at RCS is stabilized by a large incoherent tune spread (Landau damping) due to non-linearity of the space-charge force.
- It is even proposed to shorten a bunch at RCS to increase the space-charge force to achieve a stronger damping of a beam, though it sounds contrary to common belief (a longer bunch is more stable).
- Is the space-charge force really a “magic cure”?

Mysterious Simulation/Analytic Results

- During HB2014 Workshop, Kornilov and Blaskiewicz reported mysterious simulation and analytical results for beam instabilities with space-charge force.

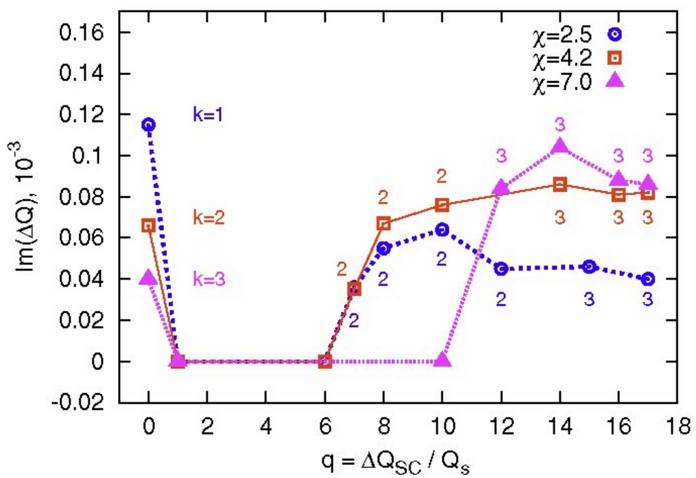


FIG. 10. Growth rates of the most unstable head-tail modes obtained in simulations for a Gaussian bunch for three different head-tail phases $\chi = \zeta\tau_b$ in a dependence on the space-charge parameter. The mode index k is given for each data point with the corresponding color.

V. Kornilov and O. Boine-Frankenheim
PRST-AB, 13, 114201 (2010)

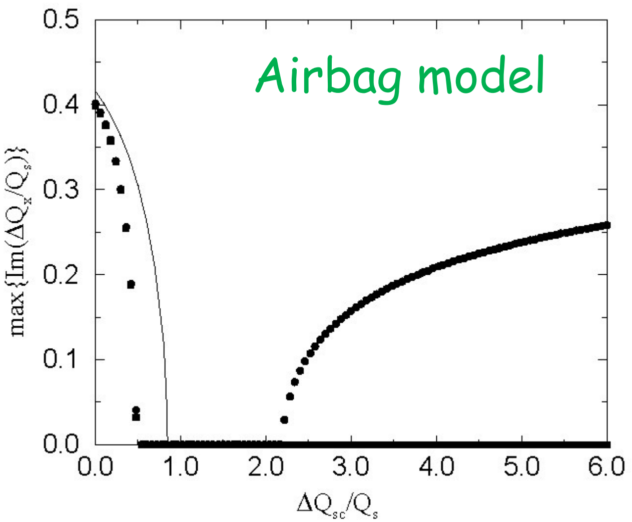


FIG. 1. Largest value of $\mathrm{Im}(\Delta Q_x/Q_s)$ as a function of $\Delta Q_{sc}/Q_s$ and $m_{\max}$ using matrix element (12). The value of W is twice the size needed to produce instability with $\Delta Q_{sc} = 0$: $m_{\max} = 1$, solid line; $m_{\max} = 5$, squares; $m_{\max} = 10$, circles.

M. Blaskiewicz
PRST-AB, 1, 044201 (1998)

Beam Instabilities with Space-Charge

- Many simulation results generally indicate that beam instability can be damped by a weak space-charge force, but the beam becomes unstable again when the space charge force is further increased.
- If the damping of beam instabilities is caused by the betatron tune spread (Landau damping) due to the non-linearity of the space-charge force,
  - A stronger space-charge force should be more effective in damping of beam instabilities.
- Why do many simulation results show the contrary?
- This mystery has not been solved for ~20 years.

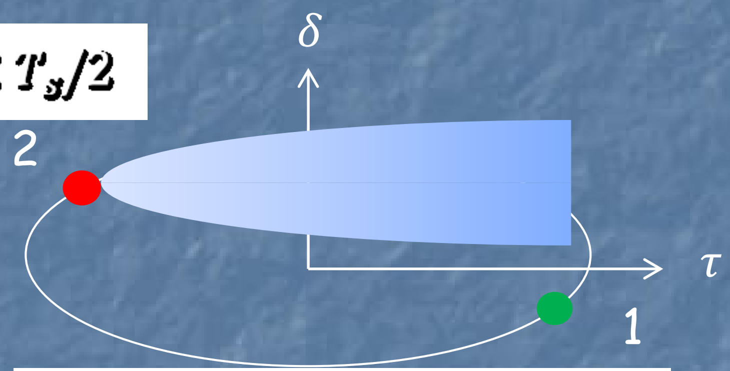
Invitation by Chao

- After the working session at HB2014, Chao has invited use to collaborate on study for effects of space-charge force on beam instabilities by modifying his famous two particle model for a strong head-tail instability.
  - That was a fascinating idea.
    - We may be able to solve the mystery by using a simple model and mathematics for this complicated phenomenon.
    - We found later though that his proposed new two particle model did not work (a pity).
  - So, it turned out that the crux of the problem is to find a suitable new two particle model which is
    - A simple expansion of the original two particle model
    - Still analytically solvable.

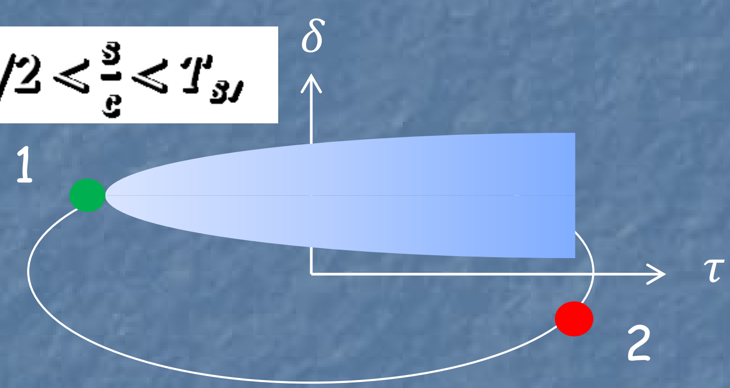
Chao's Original Two Particle Model

- Let us first review the premise and treatment of Chao's original two particle model.
  - Two macro-particles executing synchrotron and betatron oscillations.
  - Their synchrotron oscillations have equal amplitude, but opposite phases.

$$0 < \frac{s}{c} < T_s/2$$



$$T_s/2 < \frac{s}{c} < T_s,$$



$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = \frac{N r_0 W_0}{2\gamma C} y_2$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = 0$$

$Y_1 \rightleftarrows Y_2$

Total Matrix for Full Synchrotron Period

- $$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=cT_s/2} = e^{-i\omega_\beta T_s/2}\begin{bmatrix}1 & i\Upsilon\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0} \text{ for } 0 < \frac{s}{c} < T_s/2$$
  - Here

$$\Upsilon = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}$$ ⟵ Dimensionless Wake Field Strength Parameter

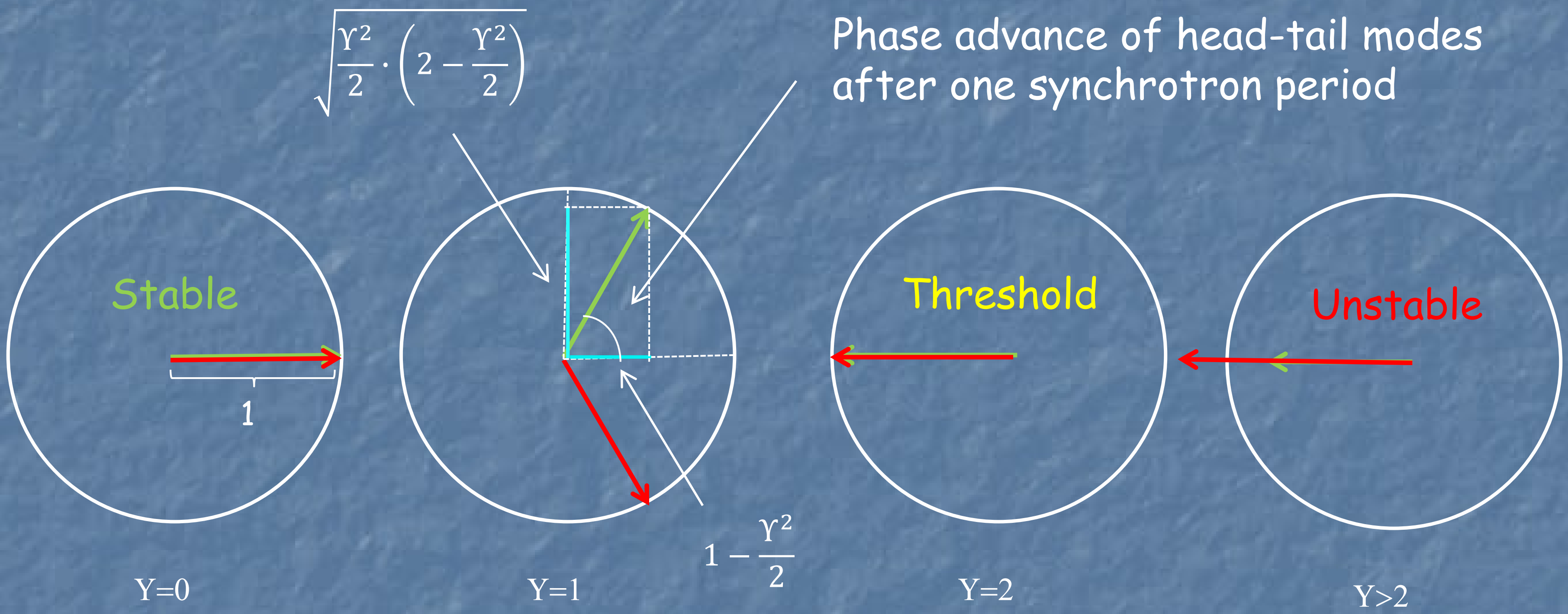
- Total Matrix
  - $$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=cT_s} = e^{-i\omega_\beta T_s}\begin{bmatrix}1 & 0\\ i\Upsilon & 1\end{bmatrix}\begin{bmatrix}1 & i\Upsilon\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0} =$$

$$e^{-i\omega_\beta T_s}\begin{bmatrix}1 & i\Upsilon\\ i\Upsilon & 1-\Upsilon^2\end{bmatrix}\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\end{bmatrix}_{s=0}.$$

Eigenvalues and Growth Rate

- The two eigenvalues are
  - $$\lambda = \begin{cases} 1 - \frac{\Upsilon^2}{2} \pm \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} & if\ \Upsilon^2 \geq 4 \\ 1 - \frac{\Upsilon^2}{2} \pm i\sqrt{\frac{\Upsilon^2}{2} \cdot \left(2 - \frac{\Upsilon^2}{2}\right)} & if\ \Upsilon^2 \leq 4 \end{cases}$$
- If $\Upsilon^2 \geq 4$, one of the solutions is unstable.
  - $$\lambda = 1 - \frac{\Upsilon^2}{2} - \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} \leq -1$$
- At the threshold value of $\Upsilon^2 = 4$, the eigenvalue λ becomes exactly minus one ($\lambda = -1$) or
  - $\lambda = e^{\pm i\pi}$.

Instability Mechanism



$$\lambda = 1-\frac{\Upsilon^2}{2} \pm i\sqrt{\frac{\Upsilon^2}{2}\cdot\left(2-\frac{\Upsilon^2}{2}\right)}$$

$$\lambda = e^{\pm i\pi}$$

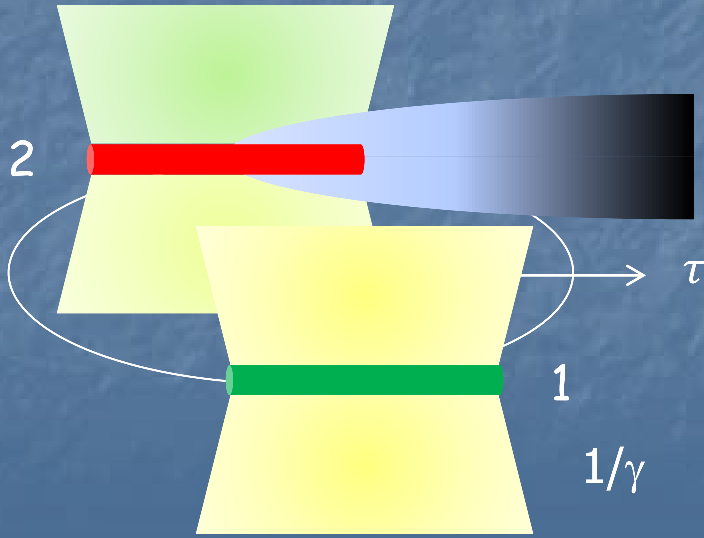
$$\lambda = 1-\frac{\Upsilon^2}{2} \pm \sqrt{\frac{\Upsilon^2}{2}\cdot\left(\frac{\Upsilon^2}{2}-2\right)}$$

Transverse Mode-Coupling Instability

- It implies that the strong head-tail instability occurs by the mode coupling between the two solutions when the difference of their phase advances over one synchrotron period becomes exactly 2π.
- The growth rate g, when $\Upsilon^2 \geq 4$, is obtained by equating
  - $|\lambda| = e^{gT_s} = \sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} + \frac{\Upsilon^2}{2} - 1.$
- The formula for the growth rate:
  - $g = \frac{1}{T_s} \log\left\{\sqrt{\frac{\Upsilon^2}{2} \cdot \left(\frac{\Upsilon^2}{2} - 2\right)} + \frac{\Upsilon^2}{2} - 1\right\}.$

New Two Particle Model with Space Charge

- Two approximations:
  - Linear Model
    - The space-charge force is linear in the relative distance between the two particles.
  - Continuous Interaction Model
    - The two particles interact continuously and coherently with a space charge force in the transverse plane.



For $0 < \frac{s}{c} < T_s/2$

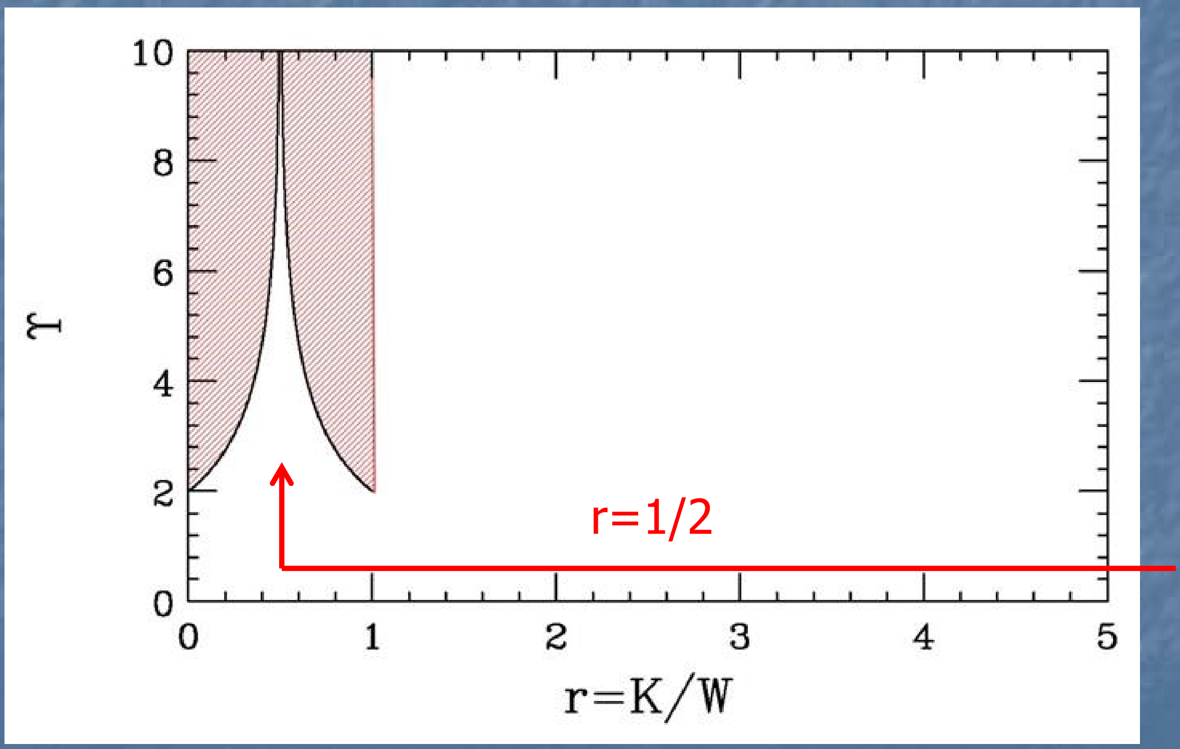
$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = K(y_1 - y_2) + W y_2$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = K(y_2 - y_1)$$

$$W = \frac{N r_0 W_0}{2\gamma C} \qquad K = \frac{N r_0}{a^2 \beta^2 \gamma^3 C}$$

Weak Space-Charge Case (W≥K)

- The two coupled equations of motion can be solved using the eigenvalue/eigenvector technique.



The stability diagram for the weak space-charge case (r=K/W≤1).

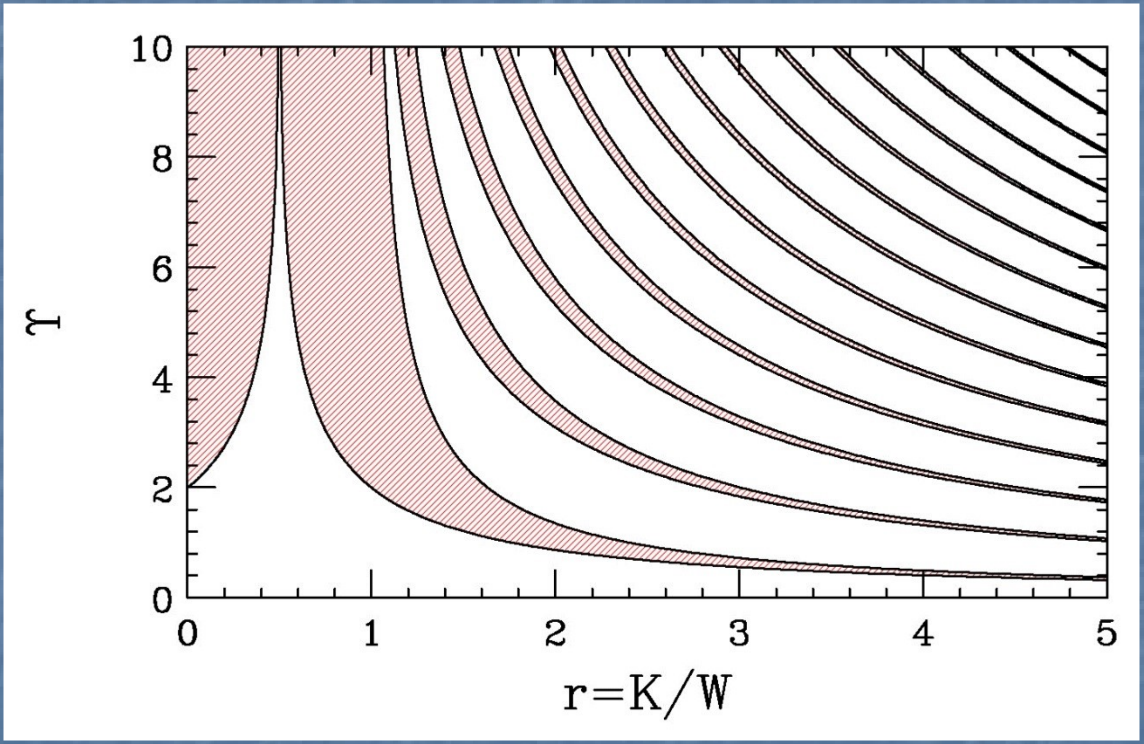
Unstable regions are shown shaded.

$$y_1'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_1 = \frac{W}{2} y_2$$

$$y_2'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_2 = -\frac{W}{2} y_1$$

Absolutely stable regardless of W

Strong Space-Charge Case (K≥W)



The stability diagram for the strong space-charge case ($r=K/W\geq1$).

The stability diagram for the weak space-charge case ($r=K/W\leq1$) is also plotted for completion.

Unstable regions are shown shaded.

Procedure to Calculate Growth rate

- For given Υ (the dimensionless wake field parameter) and $\Delta\nu_{sc}/\nu_s$ (the dimensionless space-charge parameter),

$$\mathrm{r} = \frac{K}{W} = \frac{\pi}{2\Upsilon}\left(\frac{\Delta\nu_{sc}}{\nu_s}\right)$$

r≤1

$$\mathrm{y} = 2\sqrt{r(1-r)}$$

$$\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right) \le \mathrm{y}^2$$

Yes

Stable

No

Unstable

$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1-y^2}{y^2} \cdot \frac{\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}{1-\tanh^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}.$$

r≥1

$$\mathrm{y} = 2\sqrt{r(r-1)}.$$

$$\tan^2\left(\frac{\Upsilon}{2}\mathrm{y}\right) \le \mathrm{y}^2$$

Yes

Stable

No

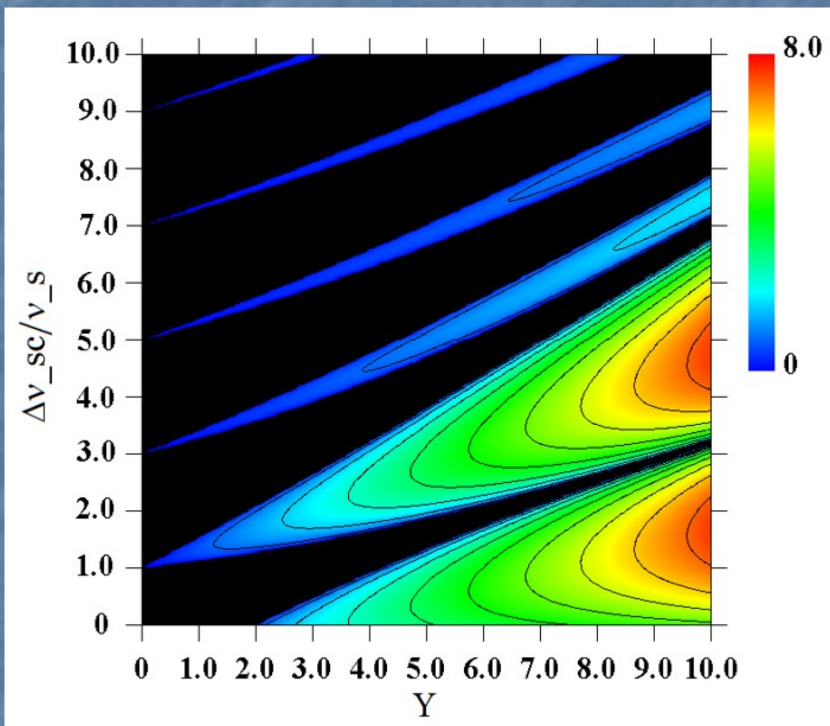
Unstable

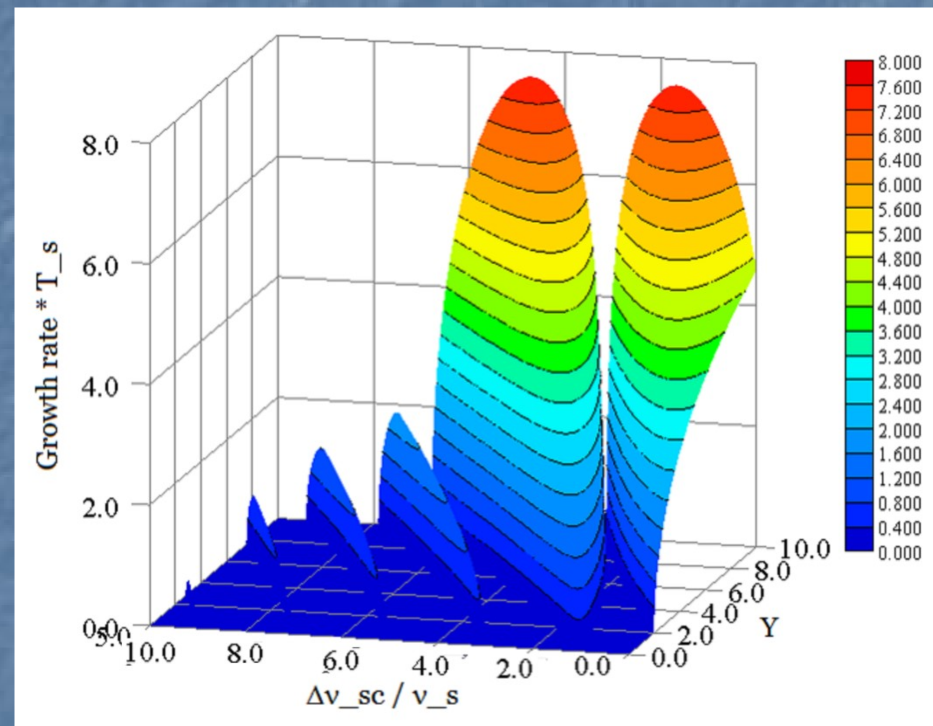
$$\frac{\Gamma^2}{2} = 2 \cdot \frac{1+y^2}{y^2} \cdot \frac{\tan^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}{1+\tan^2\left(\frac{\Upsilon}{2}\mathrm{y}\right)}.$$

Growth rate: $g = \frac{1}{T_s}\log\left\{\sqrt{\frac{\Gamma^2}{2}\cdot\left(\frac{\Gamma^2}{2}-2\right)}+\frac{\Gamma^2}{2}-1\right\}$

Contour Plots for Growth Rate

These figures are all universal !



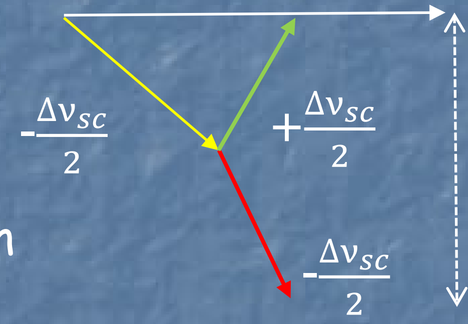


Flat contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.

3-dimensional contour plot for the growth factor $g \times T_s$ as a function of Υ and $\frac{\Delta\nu_{sc}}{\nu_s}$.

Instability Mechanism

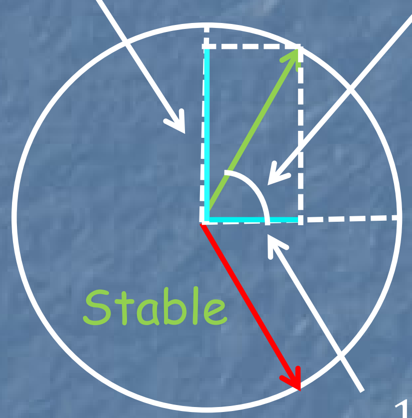
In the strong space-charge regime:



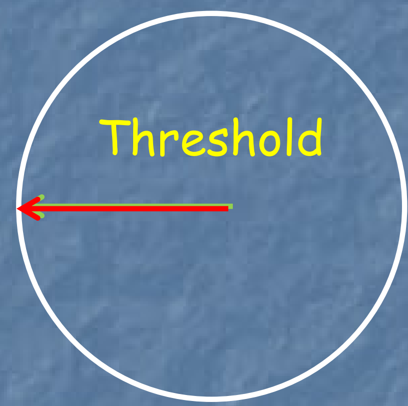
Mode-coupling when equal to odd integers times ν_s

Phase advance of head-tail mode after one synchrotron period

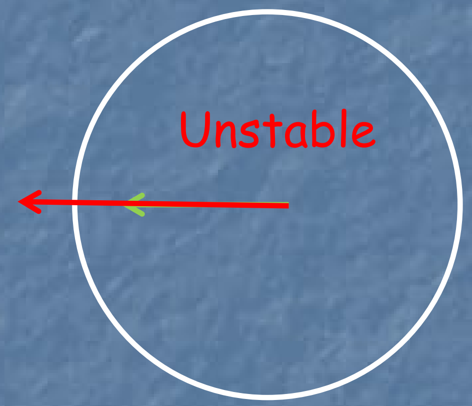
$$\sqrt{\frac{\Gamma^2}{2}\cdot\left(2-\frac{\Gamma^2}{2}\right)}$$



$$1-\frac{\Gamma^2}{2}$$



$\Gamma=2$

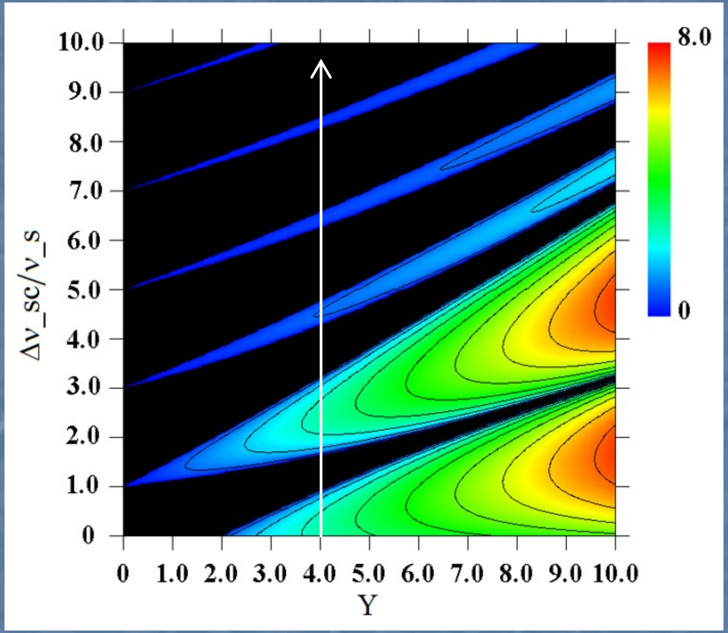


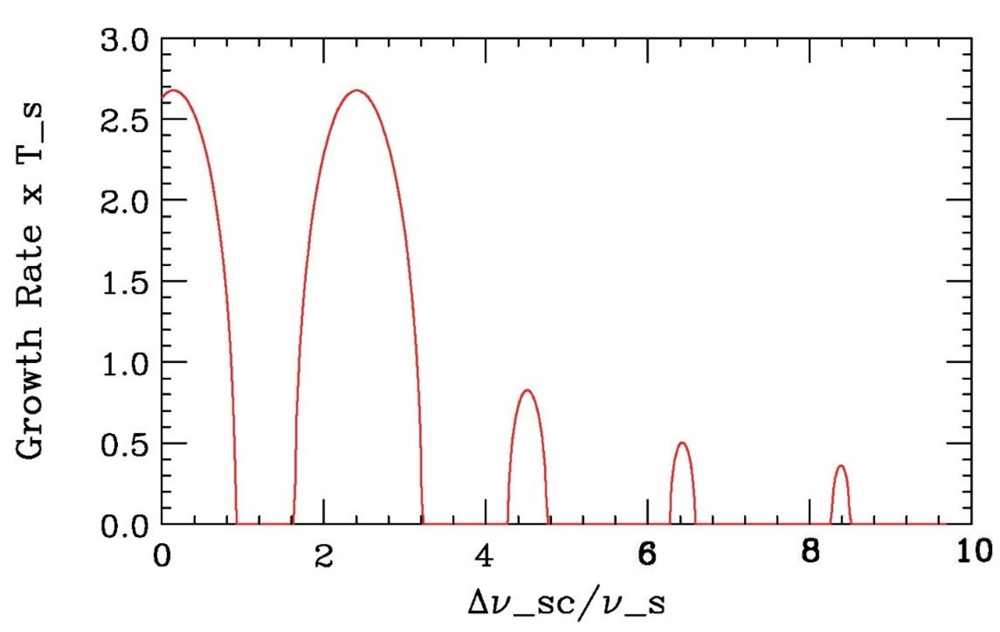
$\Gamma>2$

$$\lambda = 1-\frac{\Gamma^2}{2} \pm i\sqrt{\frac{\Gamma^2}{2}\cdot\left(2-\frac{\Gamma^2}{2}\right)}$$

Growth Rate as a Function of Space-Charge Tune Shift

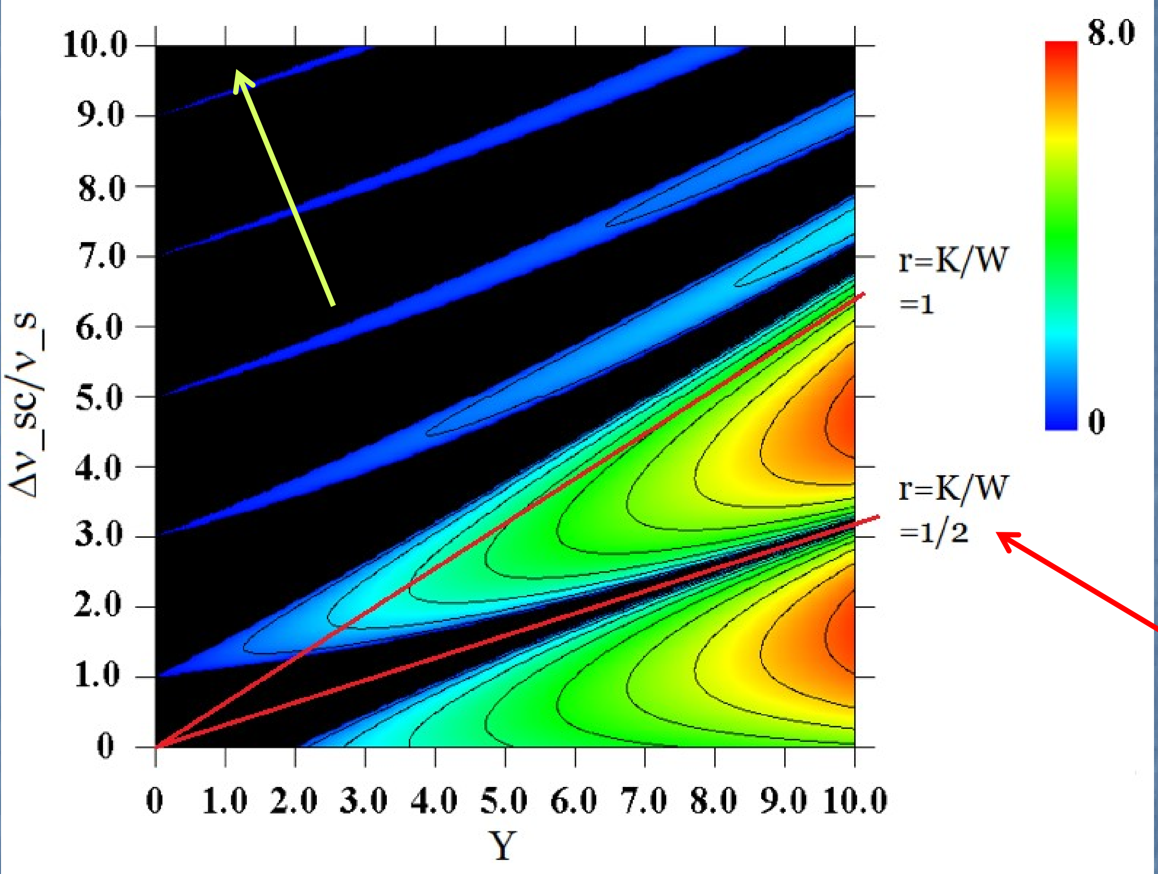
- Υ =4 case.





- It shows that the space-charge force loses its damping effect when it is too strong.
- It qualitatively reproduces typical behaviors shown in many theoretical and simulation results.

Two Cases of Absolutely Stable Coupled Motions



As the space-charge force increases, Eqs. of motion approach to those for two pendulums connected with a spring.

$$y_1'' + \left(\frac{\omega_\beta}{c}\right)^2 y_1 = K(y_1 - y_2)$$

$$y_2'' + \left(\frac{\omega_\beta}{c}\right)^2 y_2 = K(y_2 - y_1)$$

Another absolutely stable motions.

$$y_1'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_1 = \frac{W}{2} y_2$$

$$y_2'' + \left[\left(\frac{\omega_\beta}{c}\right)^2 - \frac{W}{2}\right] y_2 = -\frac{W}{2} y_1$$

Findings and Conclusions

- The present two particle model has no tune spread effect, since the space-charge force is linearized in the transverse position.
- The damping of beam instabilities with a weak space-charge force is caused by pure coherent kicks of the space-charge force in a way to partially neutralize the coherent wake field kicks.
- The damping by linear coherent kicks is unusual ?
  - No. To damp beam instabilities externally, we often use
    - Non-linear magnets such as octupoles for Landau damping by an incoherent tune spread.
    - Feedback system for linear (in the transverse displacement of a beam from the center orbit) coherent kicks to a beam.

Biased Perception

- The present model suggests that the main damping mechanism of beam instabilities with a weak space-charge force is linear coherent space-charge kicks, not the Landau damping due to the non-linearity of the space-charge force.
- However, when we study on damping of beam instabilities by a beam itself, we tend to think only Landau damping as a damping mechanism of space-charge force (because of a large tune spread).
- Further investigation of the present model and/or inclusion of more effects will help us to have a better understanding of effects of space-charge force on beam instabilities.

"The Geography of Thought" by R. Nisbett
How Asians and Westerners Think Differently… and Why

- According to this book,
  - Westerners think that the World is simple and steady.
    - It is ruled by simple laws of nature and can be described by simple models.
    - They value principles.
  - Asians think that the World is complicated and rapidly changing.
    - It is too complicated even to describe.
    - There is no law of nature, since such a law is also changing all the time.
    - They value practicality.
  - That is why westerners succeeded in creating and developing science called physics, while We Asians failed.
- Let's do it in Westerners' way to see how it works.